國立中山大學109學年度寒假轉學考招生考試試題

科目名稱：線性代數【應數系二年級】　　題號：724002
※本科目依簡章規定「不可以」使用計算機　　共1頁第1頁

1. [20%] Let

$$A = \begin{pmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 2021 & 1 \\ 0 & 0 & 0 & a \end{pmatrix}$$

   where $a, \theta \in \mathbb{R}$.

   (1) Find A^n, $n \in \mathbb{N}$.

   (2) Show that A is invertible if and only if $a \neq 0$. Find A^n, $n \in \mathbb{Z}$, if A is invertible.

2. [15%] Find the Jordan canonical form of the matrix

$$\begin{pmatrix} -4 & 2 & 10 \\ -4 & 3 & 7 \\ -3 & 1 & 7 \end{pmatrix}.$$

3. [20%] Let S be the set of solutions to the system of linear equations

$$x - 2y + z = 0$$
$$2x - 3y + z = 0.$$

   (1) Show that S is a subspace of $\mathbb{R}^3$.
   (2) Find a basis for this subspace S.
   (3) What is the dimension of S?
   (4) Let $\{v_1, v_2, \ldots, v_n\}$ be a linearly independent subset of S. What is the possible maximal value of n?

4. [15%] Let A be an invertible $n \times n$ matrix so that $A^n = I$. Show that $I - A$, where I is the $n \times n$ identity matrix, is invertible. Find $(I - A)^{-1}$.

5. [15%] Let T be the linear transform on the set $M_n(\mathbb{R})$ of $n \times n$ matrices over $\mathbb{R}$ defined as $T(A) = A^t$, the transpose of $A \in M_n(\mathbb{R})$. Find all eigenvalues of T and the corresponding eigenvectors.

6. [15%] Let V be a vector space with $\dim V = n$. Suppose that $T : V \to V$ is a linear transformation and $v \in V$ satisfying $A^{n-1}v \neq 0$ and $A^n v = 0$. Show that A admits the matrix representation

$$\begin{pmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix}$$

   with respect to some basis of V.