Differential Equations

PhD Qualifying Examination

National Sun Yat-sen University

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There are in total 100 points. You get full points of each problem only if your written reasoning is complete and the answer is correct.

Problem 1. *Let $M_n(\mathbb{R})$ be the vector space of all n-by-n real matrices. Given $A \in M_n(\mathbb{R})$ and $t \in \mathbb{R}$, the matrix exponential is defined by*

$$e^{tA} := \sum_{j=0}^{\infty} \frac{(tA)^j}{j!}.$$

Determine whether each of the following statements is true. If it is true, give a proof. If it is false, give a counterexample.

(i) (10 points) *If $A, B \in M_n(\mathbb{R})$ satisfy $AB = BA$, then $e^{t(A+B)} = e^{tA}e^{tB}$ for all $t \in \mathbb{R}$.*

(ii) (10 points) *There exist $A, B \in M_n(\mathbb{R})$ such that $e^A = e^B = \mathcal{I}_n$ but $AB \neq BA$. Here $\mathcal{I}_n \in M_n(\mathbb{R})$ is the identity matrix.*

Problem 2. *Consider the Duffing pendulum equation*

$$\ddot{x}(t) + V'(x(t)) = 0$$

with the potential $V(x) = x^4 - 2x^2 + 1$ (and thus $V'(x) = 4x^3 - 4x$).

(i) (6 points) *Show that there are three equilibria $(x, \dot{x}) = (-1, 0), (0, 0), (1, 0)$. For each equilibrium, determine whether it is locally stable or unstable.*

(ii) (6 points) *Prove that there exists a homoclinic orbit that connects the equilibrium $(x, \dot{x}) = (0, 0)$, i.e., a nonequilibrium solution $x(t)$ such that $\lim_{t\to\pm\infty} x(t) = 0$ and $\lim_{t\to\pm\infty} \dot{x}(t) = 0$.*

(iii) (8 points) *Prove that for every $p > 0$ there exists a periodic orbit with p as the minimal period and it is symmetric to the origin $(x, \dot{x}) = (0, 0)$.*

Problem 3. *Let $a \in C^0(\mathbb{R})$. Consider the Sturm–Liouville eigenvalue problem*

$$u''(x) + a(x)u(x) = \mu u(x)$$

on the unit interval $x \in (0, 1)$ with Neumann boundary conditions $u'(0) = u'(1) = 0$.

(i) (10 points) *Show that all eigenvalues are real, algebraically simple, and can be listed as $\mu_0 > \mu_1 > ... > \mu_n > ...$ such that $\lim_{n\to\infty} \mu_n = -\infty$.*

(ii) (10 points) *Show that the n-th eigenfunction $u_n(x)$ associated with μ_n possesses exactly n simple zeros in $(0, 1)$.*

Problem 4. (10 points) *Let $a, b : \mathbb{R} \to (0, \infty)$ be bounded continuous functions. Consider a solution $u \in C^1(\mathbb{R}^2, \mathbb{R})$ of*

$$a(x)u_x(x, y) + b(y)u_y(x, y) = 0.$$

Prove the existence of functions $f, g, h \in C^1(\mathbb{R})$ such that

$$u(x, y) = f(g(x) + h(y)).$$

Problem 5. *Answer the following questions.*

(i) (10 points) *Show that we can express a solution $u \in C^\infty((0, \infty) \times \mathbb{R}, \mathbb{R})$ of the linear heat equation*

$$u_t(t, x) = u_{xx}(t, x),$$

in the form

$$u(t, x) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4t}} u_0(y)\, \mathrm{d}y, \tag{1}$$

where $u_0 \in C^0(\mathbb{R})$ is any bounded function.

(ii) (10 points) *Let $u(t, x)$ be the solution expressed in (1). Show*

$$\lim_{t\to 0^+} u(t, x) = u_0(x) \quad \textit{for all } x \in \mathbb{R}.$$

Problem 6. (10 points) *Let $u \in C^2(\mathbb{R}^n)$ be a bounded function such that $\Delta u = 0$. Show that u is a constant function.*