Qualifying Exam in Differential Equations

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Solve all problems

(1) (20 points) Consider the transport equation

$$\begin{cases} \dfrac{\partial u}{\partial t} + y\dfrac{\partial u}{\partial x} = 0\,, & (t,x,y) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}\,, \\ u(0,x,y) = u_0(x,y)\,, & \end{cases}$$

where $u_0(x,y)$ is a continuous function with $u_0(x,y) = 0$ for $|x| + |y| \geq 1$.
a. Solve the equation.
b. Prove that for fixed $t_0 > 0$, $y_0 \in \mathbb{R}$,

$$\lim_{x\to\infty} u(t_0, x, y_0) = 0\,.$$

(2) (20 points) Let Ω be a open, bounded, connected subset of $\mathbb{R}^2$ and u, v be harmonic functions with $u \geq v$ on $\partial\Omega$.
a. Prove that either $u > v$ on Ω or $u = v$ on $\overline{\Omega}$.
b. Give an example to show that the conclusion in part a. fail if Ω is open, connected, but not bounded in $\mathbb{R}^2$.

(3) (20 points) Find the solution of the problem:

$$\begin{cases} u_t = u_{xx}\,, & \\ u(0,x) = 0\,, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}^+\,. \\ u(t,0) = h(t), \quad h(0) = 0\,, & \end{cases}$$

(4) (20 points) Let $\eta(t)$ be a nonnegative differentiable function on $[0, T]$ which satisfies the inequality

$$\eta'(t) \leq \phi(t)\eta(t) + \psi(t)\,,$$

where $\phi(t)$ and $\psi(t)$ are nonnegative continuous function on $[0, T]$, prove that for all $0 \leq t \leq T$

$$\eta(t) \leq e^{\int_0^t \phi(s)ds} \left[\eta(0) + \int_0^t \psi(s)ds \right] .$$

(5) (20 points) Consider the system

$$\begin{cases} \dfrac{dx_1}{dt} = x_2 - x_1(x_1^2 + x_2^2)\,, \\ \dfrac{dx_2}{dt} = -x_1 - x_2(x_1^2 + x_2^2)\,. \end{cases} \qquad (N)$$

a. Prove that $(0, 0)$ is asymptotically stable for the nonlinear system (N).
b. Let $x' = Ax$ be the linearized system of (N) around $(0, 0)$, find the matrix A and prove that $(0, 0)$ is the center for the linearized system.