All Hilbert spaces and Banach spaces are over the complex field.

1. Let l^1 denote the collection of sequences of complex scalars $\tilde{c} = (c_1, c_2, \cdots, c_n, \cdots)$, where $c_n \in \mathbb{C}$ for $n = 1, 2, \cdots$, such that $\sum_{n=1}^{\infty} |c_n| < \infty$. Please state the definition of a Banach space and show that l^1 has a Banach space structure with the norm $||\tilde{c}|| = \sum_{n=1}^{\infty} |c_n|$.

2. State the Hahn-Banach extension theorem for bounded linear functionals. Then explain how to use the real Hahn-Banach theorem to deduce the complex Hahn-Banach theorem.

3. Let $H = L^2([0, 1])$ be the space of square-summable complex-valued functions with respect to the standard Lebesgue measure over the unit interval $[0, 1]$. Let $Vf(x) = \int_0^x f(t)dt$ for any $f \in H$ and $x \in [0, 1]$. Show that V is a bounded operator on H and find its adjoint V^*.

4. State the definition of a compact operator on a Hilbert space. Then decide whether V above is compact. (Hint: There are different but equivalent ways to define compact operators. Just state any of them.)

5. Let H be the Hilbert space of square-integrable analytic functions over the unit disk in the complex plane with respect to the normalized area measure, with the natural inner product. In other words, let $dA(z)$ denote the Lebesgue measure on $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, and let
$$||f||^2 = \frac{1}{\pi} \int_{\mathbb{D}} |f(z)|^2 dA(z).$$
Let $z_0 \in \mathbb{D}$ be any point in the disk. Show that the map $f \mapsto f(z_0)$ is a bounded linear functional on H and find its norm.

6. Let H be a Hilbert space with an orthonormal basis $\{e_n\}_{n=1}^{\infty}$. A diagonal operator T is an operator such that $Te_n = c_n e_n$ for some $c_n \in \mathbb{C}$ for each $n \geq 1$. Give an example of a bounded diagonal operator whose range is not closed. Justify your answer please.

7. Let X be a Banach space. Let $E \subset X$ be a closed subspace of X and $F \subset X$ be a finite dimensional subspace of X. Show that $E + F$ is a closed subspace of X.

8. Let X be a Banach space. Please define the weak topology on X. Then show that if $A \subset X$ is a weakly compact subset of X, then A is bounded.

9. Let X be a Banach space and let $\{x_n\}_{n\geq 1}$ be a weakly convergent sequence with weak limit $\tilde{x}$. Let $\sigma_n = \frac{1}{n}(x_1 + \cdots + x_n)$ for each $n \geq 1$. Show that the sequence $\{\sigma_n\}_{n\geq 1}$ converges weakly to $\tilde{x}$.

10. Let $\mathcal{A}$ be an algebra with an involution $*$. If there are two norms $|| \cdot ||_1$ and $|| \cdot ||_2$, each making $\mathcal{A}$ into a C^*-algebra, then show that the two norms are equal.