Department of Applied Mathematics, National Sun Yat-sen University

Ph.D. Program Qualifying Exam

Functional Analysis, 2021

In the following $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

Prove any 5 of following questions (20 points for each questions)

1. Riesz representation theorem：
   Let H be a Hilbert space, if $L : H \to \mathbb{F}$ is a bounded linear functional, then there is a unique vector $h_0 \in H$, such that $L(h) = \langle h, h_0 \rangle$ for every $h \in H$.
   Moreover, $\|L\| = \|h_0\|$.
2. Two Hilbert spaces are isomorphic if and only if they have the same dimension.
3. Let H be a Hilbert space. Show that $A \in B(H)$ then $\|A\| = \|A^*\| = \|A^*A\|^{\frac{1}{2}}$.
4. Let H be a Hilbert space, suppose that T is a compact self-adjoint operator on H, then either $\pm\|T\|$ is an eigenvalue.
5. If X is a normal linear space, μ is a linear manifold in X, and $f : \mu \to \mathbb{F}$ is a bounded linear functional, then there is an F in X^* such that $F|_\mu = f$ and $\|F\| = \|f\|$.
6. If X is a normed linear space and suppose the surface S of the unit sphere in X is compact. Then X is finite dimension.
7. A normed linear space X is a Banach space if and only if every absolute convergent series in X is convergent.
8. Let M be a closed subspace of a Banach space X consider the quotient space

   $X/M = \{[x] : x \in X\}$ where $[x] = \{u : x - u = m, \text{some } m \in M\}$.

   Define the quotient norm on X/M by $\|[x]\| = \inf_{m \in M} \|x - m\|$, then X/M is a Banach space.
9. Let X and Y be Banach spaces and let $A \in B(X, Y)$. If A is a compact operator then A is completely continuous.
10. Let A be a Banach algebra with identity 1_A, then the set of invertible elements in A is open.