National Sun Yat-Sen University

Qualifying Exam -Functional Analysis

Time: Total points available: 100

Name and student number:

- There are **7** problems. You need to answer **5** of them.
- Please write down problem numbers that you would like to answer at the beginning of the answer sheet. If you answer extra problems, you would **not** receive extra points.

1. (20 points) Let X be a Banach space with norm $\|\cdot\|$.
   (a) Show that if $x_n \to x$ strongly (in X), then $x_n \to x$ weakly.
   (b) Suppose that $x_n \to x$ weakly. Would $x_n \to x$ strongly? Justify your answer.
2. (20 points) Let $\mathcal{H}$ be a Hilbert space with norm $\|\cdot\|$. Prove that if A is a compact operator on $\mathcal{H}$, then either $\pm\|A\|^2$ is an eigenvalue of A^*A.
3. (20 points) Let $\mathcal{H}$ be a Hilbert space with norm $\|\cdot\|$. Suppose that $h_n \to h$ weakly and $\|h_n\| \to \|h\|$. Show that $\|h_n - h\| \to 0$.
4. (20 points) Show that if the closed unit ball in a normed linear space X is compact, then X is finite-dimensional.
5. (20 points) Prove that if $K(x,y)$ is continuous on $[0,1]\times[0,1]$, then the operator T defined by

$$(T(f))(x) = \int_{[0,1]} K(x,y)f(y)dy$$

   (over $C[0,1]$) is compact, and find T^*.
6. (20 points) State and prove the Uniform Boundedness Principle.
7. (20 points)
   (a) Define the dual space of a Banach space X.
   (b) Let $p \geq 1$ and q satisfy $\frac{1}{p} + \frac{1}{q} = 1$. Show that the dual space of $L^p[0,1]$ is $L^q[0,1]$.