Department of Applied Mathematics, National Sun Yat-sen University

Qualifying Examination (PhD)

Functional Analysis (Fall, 2024)

Answer any <u>5</u> of the following questions.

1. Let f be a continuous function on $[0, 1]$. Suppose all its moment
$$m_n(f) = \int_0^1 f(x)x^n\, dx = 0, \quad n = 0, 1, 2, \ldots.$$
Prove that $f = 0$ on $[0, 1]$.

2. Let $x_n \to x$ in the weak topology of a Hilbert space H with $\|x_n\| \to \|x\|$. Show that $\lim_{n\to\infty} \|x - x_n\| = 0$.

3. Prove that the sequence space
$$\ell_p = \{(x_n) : \sum_{n=1}^{\infty} |x_n|^p < \infty\}$$
equipped with the p-norm
$$\|(x_n)\|_p = (\sum_{n=1}^{\infty} |x_n|^p)^{1/p}$$
is not a Hilbert space unless $p = 2$.

4. Prove that the range of a compact linear operator $T : E \to F$ between Banach spaces must be separable.

5. Let φ be a linear functional of the real Hilbert space $L^2[0, 2\pi]$ defined by
$$\varphi(f) = \int_0^{2\pi} f(2\pi - x)dx.$$
Prove that $\|\varphi\| = \sqrt{2\pi}$, and find an g in $L^2[0, 2\pi]$ such that
$$\varphi(f) = \int_0^{2\pi} f(x)g(x)dx, \quad \forall f \in L^2[0, 2\pi].$$

6. Prove that any subset I of the Banach algebra $C(X)$ of the form
$$I = \{f \in C(X) : f|_Y = 0\}$$
for a closed subset Y of the compact Hausdorff space X is a closed ideal. Is the converse also true?

7. Let x be an invertible element of a Banach algebra A. Prove that
$$\sigma(x^{-1}) = \{\lambda^{-1} : \lambda \in \sigma(x)\}.$$

8. Let x be a normal element of a C*-algebra, i.e. $x^*x = xx^*$. Prove that $\|x\| = r(x)$, where $r(x) = \lim_{n\to\infty} \|x^n\|^{1/n}$ is the spectral radius of x.

End of Paper