Ph.D. Qualifying Examination

**Matrix Analysis**

September 1, 2022

Please write down all the details of your computation and answers.

[1]. (10%) Given an $n \times n$ real matrix $A$ which has distinct eigenvalues.

Show that the eigenvectors associated with the distinct eigenvalue are linearly independent.

[2]. (5%+5%+5%) Given a matrix

$$A = \begin{bmatrix} 1 & 1 & 2 & 1 & 1 \\ 1 & 5 & 2 & 9 & 1 \\ 2 & 2 & 5 & 4 & 3 \\ 1 & 9 & 4 & 21 & 3 \\ 1 & 1 & 3 & 3 & 3 \end{bmatrix}$$

(a) Compute the LU factorization of $A$.

(b) Compute the determinant of $A$.

(c) Find a basis of null space of $A$.

[3]. (15%) State and prove Schur's unitary triangularization theorem.

[4]. (15%) Let

$$A = \begin{bmatrix} 2 & 0 & 0 & -1 \\ 3 & 2 & 0 & 2 \\ 1 & 0 & 2 & 4 \\ 1 & 0 & 0 & 4 \end{bmatrix}$$

Find a matrix $P$ such that $P^{-1}AP$ is in Jordan canonical form.

[5]. (10%) Let vectors $\mathbf{x} = [0, 2, 0, 2]^T$ and $\mathbf{y} = [2, 0, 2, 0]^T$. Find an orthogonal matrix $B$ such that $B\mathbf{x} = \mathbf{y}$.

[6]. (10%+10%) Given a $5 \times 4$ real matrix $A \in \mathbb{R}^{5\times 4}$.

(a) State one of methods to find the QR factorization of $A$ in detail.

(b) If $rank(A) = 4$, show that there exist a unique $Q \in \mathbb{R}^{5\times 4}$ with orthonormal columns and a unique upper triangular matrix $R \in \mathbb{R}^{4\times 4}$ with positive diagonal elements such that $A = QR$.

[7]. (15%) Let $A \in \mathbb{R}^{n\times n}$ be nonnegative. The spectral radius $\rho(A) = \max\{|\lambda| : \lambda \text{ is an eigenvalue of } A\}$.

Prove or disprove that if $A$ has a positive eigenvector, its corresponding eigenvalue is $\rho(A)$.

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