Ph.D. Qualifying Examination
Matrix Theory
Sep. 5, 2024

Please write down all the detail of your computations and proofs.

1. (10%) Let A be an $n \times n$ Hermitian matrix. Prove that there exist real numbers $\{c_i\}_{i=1}^n$ and an orthonormal set of column vectors $\{\mathbf{v}_i\}_{i=1}^n$ such that
$$A = \sum_{i=1}^n c_i \mathbf{v}_i \mathbf{v}_i^T.$$

2. (10%) Let A be an $n \times n$ Hermitian positive semidefinite matrix. Prove that there exists a unique Hermitian positive semidefinite square root matrix B of A such that $B^2 = A$.

3. (15%) Consider the Fibonacci sequence $y_{n+1} = y_n + y_{n-1}$ for $n = 1, 2, \cdots$ with $y_0 = 0$ and $y_1 = 1$. Find the matrix A satisfying
$$\begin{bmatrix} y_{n+1} \\ y_n \end{bmatrix} = A \begin{bmatrix} y_n \\ y_{n-1} \end{bmatrix}.$$
Diagonalize A and use it to obtain the explicit exact formula of y_n.

4. (15%) Let $\|\cdot\|_v$ be an vector norm on $\mathbf{C}^n$. Prove that
$$|||A|||_v = \sup_{\mathbf{x} \neq \mathbf{0}} \frac{\|A\mathbf{x}\|_v}{\|\mathbf{x}\|_v}$$
is a matrix norm and is compatible with $\|\cdot\|_v$.

5. (15%) Prove that all irreducibly diagonally dominant matrices are nonsingular.

6. (15%) Let A be an $n \times n$ diagonalizable positive matrix with each column sum 1. What is the limit of the sequence $\mathbf{x}_{k+1} = A\mathbf{x}_k$ if $\mathbf{x}_0$ is nonnegative with sum of entries 1? What is the limit of A^k as $k \to \infty$? Prove your answers.

7. (20%) Let A be a real $m \times n$ matrix, $\mathcal{N}(A)$ and $\mathfrak{R}(A)$ be its null and range spaces, respectively. Prove that the orthogonal direct sums

   1) $\mathbb{R}^n = \mathcal{N}(A) \overset{\perp}{\oplus} \mathfrak{R}(A^T)$, *2)* $\mathbb{R}^m = \mathcal{N}(A^T) \overset{\perp}{\oplus} \mathfrak{R}(A)$,

   3) the restriction $A|_{\mathfrak{R}(A^T)} : \mathfrak{R}(A^T) \to \mathfrak{R}(A)$ is isomorphic,

   4) $\mathcal{N}(AA^T) = \mathcal{N}(A^T)$, *5)* $\mathfrak{R}(AA^T) = \mathfrak{R}(A)$.