PH.D QUALIFYING EXAM : REAL ANALYSIS 2021 SPRING

1.(10%) Assume $f(x)$ is a real-valued measurable function. Prove or disprove that $|f(x)|^2$ is also a measurable function.

2.(15%) Suppose $f(x)$ is a differentiable function on $\mathbb{R}^1$. Is it true that $f'(x)$ is a measurable function ? Prove or disprove it.

3.(10%) Let Z be a measure zero set in $\mathbb{R}^1$. Show that the set $\{x^2 : x \in Z\}$ has measure zero.

4.(15%) If $p > 0$, $\int_E |f_k - f|^p \to 0$, and $\int_E |f_k|^p \leq M$ for all k, show that $\int_E |f|^p \leq M$.

5.

(1).(10%) Suppose $f \in L^s(\mathbb{R}^n) \cap L^t(\mathbb{R}^n)$ for some $1 < s < t < \infty$. Will f belong to $L^p(\mathbb{R}^n)$ for $s < p < t$? Prove or disprove your result.

(2).(15%) For $0 < p < 1$. Show that $||f+g+h||_p \leq 3^{\frac{1-p}{p}}(||f||_p + ||g||_p + ||h||_p)$ and $3^{\frac{1-p}{p}}$ is best possible.

6.

(1).(10%) Let

$$f(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

Find the value of its maximal function at $x = -2$, i.e. what is the value of $f^*(-2)$, where f^* denotes the Hardy-Littlewood maximal function of f.

(2).(15%) Given $g \in L^2(\mathbb{R}^n)$, show that $||g^*||_2 \leq c||g||_2$ for some constant c only depending on the dimension n.