Department of Applied Mathematics
National Sun Yat-sen University

Ph. D. Qualifying Examination

Spring, 2022.

Real Analysis

Answer all 10 questions below, each of them carries 10 points out of 100 points.

(1) Let $\{a_\lambda : \lambda \in \Lambda\}$ be a family of real numbers, where the index set Λ might not be countable.

(a) Define the notion of the convergence of the sum $\sum_\lambda a_\lambda$.

(b) When the sum is finite, show that $a_\lambda = 0$ for all but at most countably many λ's.

(2) Prove that if a subset A of $[0,1]$ has measure zero then

$$A^3 = \{x^3 : x \in A\}$$

has measure zero, too. In general, if f is a continuously differentiable function on $[0,1]$, can you conclude again that

$$f(A) = \{f(x) : x \in A\}$$

has measure zero? Justify your answer.

(3) Let $I = [0,1]$. Let $m^*(E)$ and $m_*(E)$ be the outer and inner (Legesgue) measures of $E \subseteq \mathbb{R}$, respectively. In other words,

$$m^*(E) = \inf\{m(O) : E \subseteq O \text{ and } O \text{ is an open set in } I\},$$

$$m_*(E) = \sup\{m(K) : K \subseteq E \text{ and } K \text{ is a compact set in } I\}.$$

Show that

(a) $m^*(A) + m_*(A) = 1$ for any subset of I.

(b) E is Lebesgue measurable if and only if $m^*(E) = m_*(E)$.

(c) E is Lebesgue measurable if and only if

$$1 = m^*(E) + m^*(I \setminus E).$$

(4) Let $f : [0,1] \longrightarrow \mathbb{R}$ be continuous and of bounded variation. Assume that f is absolutely continuous on $[\epsilon, 1]$ for each $\epsilon > 0$. Show that f is absolutely continuous on $[0,1]$.

(5) Let $f_n \to f$ on $[0,1]$ in the following sense: for every x in $[0,1]$, if $x_n \to x$, then $f_n(x_n) \to f(x)$. Show that f is continuous if all f_n are continuous.
[Hint: Show that there are $1 \leq n_1 < n_2 < \cdots$ such that $f_{n_k}(x_k) \to f(x)$. Then consider the sequence $(y_n) = (\underbrace{x_1, \ldots, x_1}_{n_1\text{-terms}}, \underbrace{x_2, \ldots, x_2}_{(n_2-n_1)\text{-terms}}, \underbrace{x_3, \ldots, x_3}_{(n_3-n_2)\text{-terms}}, \ldots)$.]

(6) Let $\{G_n\}_n$ be a sequence of non-empty open sets in $[0,1]$ with the Lebesgue measures $m(G_n) \leq 1/2^n$ for $n = 1, 2, \ldots$. Let

$$f(x) = \sum_{n=1}^{\infty} m(G_n \cap [0,x]), \quad 0 \leq x \leq 1.$$

Show that f is continuous, non-decreasing, and that $f'(x) = +\infty$ for all x in $\cap_{n=1}^{\infty} G_n$.

(7) Let λ and μ be two positive Borel measures on $\mathbb{R}^n$ such that λ and μ are finite on compact sets and for every continuous function f on $\mathbb{R}^n$ with compact support,

$$\int_{\mathbb{R}^n} f d\lambda = \int_{\mathbb{R}^n} f d\mu.$$

Show that $\lambda = \mu$.

(8) Let μ be a finite real measure on a measurable space $(X, \mathcal{B})$. In particular,

$$-\infty < \mu(E) < +\infty, \qquad \text{for all } E \in \mathcal{B}.$$

We call A a *positive set* (resp. *negative set*) of μ if $\mu(E) \geq 0$ (resp. $\mu(E) \leq 0$) for all measurable subset E of A.

(a) Let

$$b = \inf\{\mu(E) : E \text{ is a negative set of } \mu\}.$$

Prove that there exist a $B \in \mathcal{B}$ such that $\mu(B) = b$, and B is a negative set of μ.

(b) Let $A = X \setminus B$. Prove that A is a positive set of μ.
[Remark. We call $X = A \cup B$ the Hahn decomposition of X and $\mu = \mu_+ - \mu_-$, where $\mu_+(E) = \mu(E \cup A)$ and $\mu_-(E) = \mu(E \cup B)$, the Jordan decomposition of μ.]

(9) Let f be a real-valued integrable function on $[0,1]$.

(a) Suppose all its moments

$$m_n(f) = \int_0^1 f(x)x^n \, dx = 0, \quad n = 0, 1, 2, \ldots.$$

Prove that $f = 0$ a.e. on $[0,1]$.

(b) Suppose

$$\int_0^x f(t) \, dt = 0$$

for all x in $[0,1]$. Prove that $f = 0$ a.e. on $[0,1]$.

(10) For $0 < \alpha < 1$, let

$$\mathcal{M}_\alpha = \{E \subseteq [0,1] : E \text{ is measurable and } m(E) = \alpha\}.$$

Let $f \in L^1[0,1]$. If

$$\int_E f \, dx = 0, \quad \text{for each } E \text{ in } \mathcal{M}_\alpha,$$

show that $f = 0$ a.e. on $[0,1]$.

[Hint: Consider the sets on which $f > 0$, $f < 0$ and $f = 0$. Show that the conclusion is true if $0 < \alpha \leq 1/2$. If $1/2 < \alpha < 1$, show that $\int_E f \, dx = 0$, for all E in $\mathcal{M}_{1-\alpha}$.]