Qualified Examination: Real Analysis
September 2022

1.(10%) Show that if E_1 and E_2 are measurable, then

$$m(E_1 \cup E_2) + m(E_1 \cap E_2) = m(E_1) + m(E_2).$$

2.(15%) If $\{f_n\}$ is a sequence of continuous functions on $[0,1]$ such that $0 \leq f_n \leq 1$ and such that $f_n(x) \to 0$ as $n \to \infty$ for every $x \in [0,1]$, show that

$$\lim_{n\to\infty} \int_0^1 f_n(x)\, dx = 0.$$

3.(15%) Let f be an increasing function on the open interval I. For $x_0 \in I$ show that f is continuous at x_0 if and only if there are sequences $\{a_n\}$ and $\{b_n\}$ in I such that for each n, $a_n < x_0 < b_n$, and $\lim\limits_{n\to\infty}[f(b_n) - f(a_n)] = 0$.

4.(15%) Let f be a bounded measurable function on a set of finite measure E. Assume g is bounded and $f = g$ a.e. on E. Show that $\int_E f = \int_E g$.

5.(15%) Let $\{f_n\}$ be a sequence of nonnegative measurable functions on E that converges pointwise on E to f . Suppose $f_n \leq f$ on E for each n. Show that

$$\lim_{n\to\infty} \int_E f_n = \int_E f.$$

6.(15%) Suppose ϕ is a real function on R. Prove that ϕ is convex if and only if

$$\phi\left(\int_0^1 f(x)\, dx\right) \leq \int_0^1 \phi(f)\, dx$$

for every real bounded measurable f.

7.(15%) (Riemann-Lebesgue) Show that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f(x) \cos nx \, dx = 0,$$

when f is (i) continuously differentiable and (ii) Lebesque integrable over $(-\infty, \infty)$.