Qualified Examination: Real Analysis
Spring 2023

1. (10%) Let E be a set in $\mathbf{R^n}$. Prove or disprove that E is measurable in $\mathbf{R^n}$ if and only if for all $\epsilon > 0$ there exist an open set $G_\epsilon \subset \mathbf{R^n}$ and a closed set $F_\epsilon \subset \mathbf{R^n}$ such that $F_\epsilon \subset E \subset G_\epsilon$ and $\mu_n^*(G_\epsilon \setminus F_\epsilon) < \epsilon$.

2. (a) (10%) State and prove Lusin's Theorem for a real-valued measurable function whose domain has finite measure in $\mathbf{R}$.

   (b) (5%) Can we generalize Lusin's theorem for a real-valued measurable function defined on $\mathbf{R}$? Justify your answer.

3. (a) (10%) Suppose $\{f_n\}$ is a sequence of nonnegative, measurable functions in $\mathbf{R}$. Show that
$$\int_{\mathbf{R}} \liminf f_n \leq \liminf \int_{\mathbf{R}} f_n.$$

   (b) (5%) Can we drop out the nonnegativity of f_n in (a)? Justify your answer.

4. Let f be integrable over $\mathbf{R}$.

   (a) (5%) Show that for all $\epsilon > 0$, there is a continuous function f_ϵ on $\mathbf{R}$ which vanishes outside a bounded set and
$$\int_{\mathbf{R}} |f - f_\epsilon| < \epsilon.$$

   (b) (10%) Let g be a bounded measurable function on $\mathbf{R}$. Show that
$$\lim_{t \to 0} \int_{\mathbf{R}} g(x)[f(x) - f(x+t)]dx = 0.$$

5. (10%) Let g be integrable over $[a, b]$ and define f on $[a, b]$ by
$$f(x) = \int_a^x g$$
for all $x \in [a, b]$. Show that f is differentiable almost everywhere on (a, b).

6. Let $E \subset \mathbf{R^n}$ be a measurable set and $1 \leq p < \infty$. Suppose $\{f_n\}$ is a sequence in $L^p(E)$ that converges pointwise a.e. on E to the function $f \in L^p(E)$.

   (a) (5%) Show that if $f_n \to f$ in $L^p(E)$, then $||f_n||_{L^p(E)} \to ||f||_{L^p(E)}$ as $n \to \infty$.

   (b) (10%) Prove or disprove that if $||f_n||_{L^p(E)} \to ||f||_{L^p(E)}$ as $n \to \infty$, then $f_n \to f$ in $L^p(E)$.

7. (10%) Let $1 < p < \infty$. Suppose T is a bounded linear functional on $L^p[0,1]$. Show that there is a function $g \in L^{\frac{p}{p-1}}[0,1]$ such that

$$T(f) = \int_{[0,1]} g \cdot f \text{ for all } f \in L^p[0,1].$$

8. (10%) Let f be of bounded variation on $[0,1]$. Show that there is an absolutely continuous function g on $[0,1]$, and a function h on $[0,1]$ that is of bounded variation and has $h' = 0$ a.e. on $[0,1]$, for which $f = g + h$ on $[0,1]$. Then show that this decomposition is unique except for addition of constants.