National Sun Yat-Sen University

Qualifying Exam - Real Analysis

Time: Total points available: 100

Name and student number:

- Unless otherwise stated, all integral signs stands for Lebesgue integrals.
- $\mathbb{Q}$ is the set of rational numbers, and $\mathbb{R}$ is the set of real numbers.
- For a fixed $1 \le p < \infty$, $\|f\|_p := \left(\int_0^1 |f(x)|^p dx\right)^{1/p}$ and $L^p([0,1]) := \{f : \|f\|_p < \infty\}$.

1. (10 points) Let $f(x) = \begin{cases} 1 & \text{if} \quad x \in \mathbb{Q} \cap [0,1]; \\ 0 & \text{if} \quad x \in \mathbb{Q}^c \cap [0,1]. \end{cases}$
   (a) (5 points) Is f Riemann integrable? If so, find the Riemann integral $\int_0^1 f(x)dx$. Justify your answer.
   (b) (5 points) Is f Lebesgue integrable? If so, find the Lebesgue integral $\int_0^1 f(x)dx$. Justify your answer.

2. (20 points) Let $f_n(x) = \begin{cases} \frac{1}{n} & \text{if} \quad x \in (0,n); \\ 0 & \text{otherwise} \end{cases}$ and $f(x) = 0$ for all $x \in \mathbb{R}$.
   (a) (10 points) Prove that $f_n \to f$ uniformly on $[0,\infty)$.
   (b) (5 points) For each fixed n, compute $\int_0^\infty f_n(x)dx$.
   (c) (5 points) Is $\lim_{n\to\infty} \int_0^\infty f_n(x)dx = \int_0^\infty \lim_{n\to\infty} f_n(x)dx$? If not, why?

3. (20 points) Let $1 < p < q < \infty$ be fixed real numbers.
   (a) (10 points) Prove that $L^q([0,1]) \subseteq L^p([0,1])$.
   (b) (10 points) Is the above assertion still valid if the domain is $\mathbb{R}$? Prove or disprove $L^q(\mathbb{R}) \subseteq L^p(\mathbb{R})$.

4. (10 points) Let $\{f_n\} \subset L^p([0,1])$. Suppose that $\|f_n - f\|_p \to 0$. Show that $f_n \to f$ in measure.

5. (10 points) Let $\{f_n\}$ be a sequence of nonnegative measurable functions defined on $[0,1]$. Suppose that $f_n \le 10$ for all n and $f_n(x) \to 7$ for almost all $x \in [0,1]$. Find $\lim_{n\to\infty} \int_0^1 f_n(x)dx$.

6. (20 points) Let f be the Cantor-Lebesgue function (see Appendix below for a brief recall) defined on $[0,1]$ and let $\mathbf{C}$ denote the Cantor Set (in $[0,1]$).
   (a) (5 points) For $x \in [0,1] \setminus \mathbf{C}$, find $f'(x)$.
   (b) (5 points) Find $\int_0^1 f'(x)dx$.

(c) (5 points) Does $\int_0^1 f'(x)dx = f(1) - f(0)$? If not, why?

(d) (5 points) Show that $f(1/4) = 1/3$.

7. (10 points) Suppose that f is nonnegative. Show that if $\int_0^1 f(x)dx = 0$ then $f = 0$ for almost all $x \in [0,1]$.

1 Appendix

In this appendix, we recall the Cantor set and the Cantor-Lebesgue function.

Consider the closed interval $[0,1]$. We begin by subdividing $[0,1]$ into thirds and removing the middle third. Then we repeat the same procedure to the remaining intervals; that is, we subdivide the intervals $[0, \frac{1}{3}]$ and $[\frac{2}{3}, 1]$ into thirds and remove their middle thirds. Repeating this process, we will reach the Cantor set $\mathbf{C}$.

Let C_k to denote the union of the intervals left at the k-th stage of the construction of the Cantor set. Define $D_k := [0,1] \backslash C_k$. Then D_k consists of the $2^k - 1$ intervals I_j^k (ordered from left to right) removed in the first k stages of constructions of the Cantor set. Let f_k be the continuous function on $[0,1]$ which satisfies $f_k(0) = 0$, $f_k(1) = 1$, $f_k(x) = j2^{-k}$ on I_j^k, $j = 1, \ldots, 2^k - 1$, and which is linear on each interval of C_k. Finally, the Cantor-Lebesgue function f is defined to be the uniform limit of f_k, i.e., $f := \lim_{k \to \infty} f_k$.