PH.D QUALIFYING EXAM : REAL ANALYSIS 2020 FALL

1.(10%) Let $f(x,y)$ be a function defined on $0 \leq x, y \leq 1$, which is continuous in each variable separately. Show that f is a measurable function.

2.(15%) Suppose $f : \mathbb{R} \to \mathbb{R}$ is uniformly continuous. Show that f must be bounded by a linear function, i.e, there exist constants A, B such that $|f(x)| \leq A + B|x|$ for all x.

3.(15%) Suppose the function $K(x,y)$ satisfies that for all $s, t \geq 0$ we have $K(s,t) \geq 0$ and $K(\lambda s, \lambda t) = \lambda^{-1} K(s,t)$ for all $\lambda > 0$. Now assume that $\int_0^\infty t^{-\frac{1}{p}} K(1,t)dt = C < \infty$ for some $1 \leq p \leq \infty$. Show that if $Tf(s) = \int_0^\infty f(t)K(s,t)dt$, then $||Tf||_p \leq C||f||_p$.

4.(15%) Let $f(x) \geq 0$ for all $x \in [0, \infty)$ and $\int_0^\infty f(x) < \infty$. Show that

$$\lim_{n\to\infty} \frac{1}{n} \int_0^n x f(x) dx = 0.$$

5.(15%) Given $f \in L^p(\mathbb{R}^1)$ for some $1 \leq p < \infty$. Given $t \in \mathbb{R}^1$, define $f_t(x) := f(x-t)$. Show that $\lim_{t\to 0} ||f_t - f||_p = 0$.

6.(15%) We say ϕ is convex in (a,b) if and only if $\phi(\frac{p_1 x_1 + p_2 x_2}{p_1 + p_2}) \leq \frac{p_1 \phi(x_1) + p_2 \phi(x_2)}{p_1 + p_2}$ for any $p_1, p_2 > 0$. Use this definition to prove that

$$\phi(\frac{p_1 x_1 + p_2 x_2 + p_3 x_3}{p_1 + p_2 + p_3}) \leq \frac{p_1 \phi(x_1) + p_2 \phi(x_2) + p_3 \phi(x_3)}{p_1 + p_2 + p_3}$$

for any $p_1, p_2, p_3 > 0$.

7.(15%) Suppose μ and ν are two positive finite measures on the measurable space (X, Σ). Show that there exists a measurable function f on X such that for all $E \in \Sigma$ we have $\int_E (1-f) d\mu = \int_E f d\nu$.